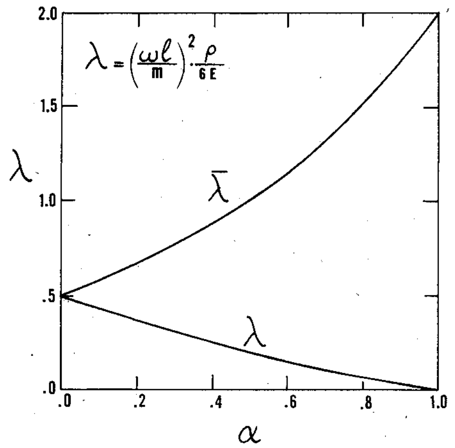

Fig. 2 Conjugate frequencies as function of α.

(2) is $D_m = 0$. But

$$D_m = (A_1 + A_2)\delta_1 D_{m-1} - A_2^2\delta_2^2 D_{m-2} \tag{3}$$

where: D_m is the determinant of Eq. (2), which has order m; D_{m-1} is the determinant of the matrix obtained by eliminating the first row and first column; and D_{m-2} is the determinant of the matrix obtained by eliminating the first two rows and columns. Based on the recursive relation (3), we can derive the following results: a) the middle frequency of an odd rod remains unchanged with respect to changes in the cross sectional areas; and b) conjugacy of the remaining frequencies. In order to prove these results, we prove the following lemmas.

Lemma a

If m is odd, δ_1 can be factored out of D_m.

Proof. Let it be assumed true for $m-2$. Then,

$$D_{m-2} = \delta_1 D'_{m-2} \tag{4}$$

Recalling relation (3), we observe that

$$D_m = \delta_1[(A_1 + A_2)D_{m-1} - A_2^2\delta_2^2 D'_{m-2}] \tag{5}$$

The result is, therefore, also true for m, which is seen to be odd since $m-2$ is odd. But $D_1 = A_1\delta_1$, hence the proof by induction is complete. Now it becomes simple to prove the initial statement. In fact, if m is odd, the equation $D_m = 0$ admits the solution $\delta_1 = 0$ since, according to lemma a, δ_1 can be factored out. Consequently, the frequency $\omega = (m/\ell)(3E/\rho)^{1/2}$ does not depend on any particular design. That this frequency is actually the middle one is a consequence of result b.

Lemma b

D_m is dimensionally homogeneous in δ_1 and δ_2. Moreover, δ_1 and δ_2 have only even powers, with the exception of the common factor δ_1 in case m is odd.

Proof: let the statement be true for D_{m-1} and D_{m-2}. We consider two cases: 1) $m-2$ is odd: recalling Eq. (3), we can write

$$D_m = \delta_1[(A_1 + A_2)D_{m-1} - A_2^2\delta_2^2 D'_{m-2}] \tag{6}$$

Since the assertion is assumed true for m-1 and $m-2$, D_{m-1} and D'_{m-2} have only even powers in δ_1 and δ_2. Except for the common factor δ_1, therefore, D_m will have only even powers in δ_1 and δ_2 too.

2) $m-2$ is even: once again, recalling Eq. (3), we can write

$$D_m = (A_1 + A_2)\delta_1^2 D'_{m-1} - A_2^2\delta_2^2 D_{m-2} \tag{7}$$

Using the same argument as item (1), we conclude that D_m has only even powers in δ_1 and δ_2. Since $D_1 = A_1\delta_1$ and $D_2 = (A_1 + A_2)\delta_1^2 A_2 - A_2^2\delta_2^2$, the proof is complete.

Consider now, the characteristic equation $D_m = 0$. We can divide both sides by $\delta_2^m \geq 1$ and eliminate the root $\delta_1 = 0$ in case m is odd. In view of lemma b, the characteristic equation can be written as

$$a_r V^r + a_{r-2}V^{r-2} + \dots + a_0 = 0 \tag{8}$$

where r is even and $V = \delta_1/\delta_2$.

It becomes clear from the structure of Eq. (8) that if α is a root of the equation, so is $-\alpha$. Recalling the definitions of δ_1 and δ_2, we have

$$\omega_j^2 = \frac{6Em^2}{\rho\ell^2}\left(\frac{1-\alpha}{2+\alpha}\right)$$
$$\bar{\omega}_j^2 = \frac{6Em^2}{\rho\ell^2}\left(\frac{1+\alpha}{2-\alpha}\right) \tag{9}$$

As one can see, for every value of α there are two frequencies interrelated by Eqs. (9). We define conjugate frequencies as the pair related by Eqs. (9). Figure 2 shows the two branches given in Eqs. (9) as a function of α. One can see that if m is odd, and therefore the invariant frequency $\omega = (m/\ell)(3E/\rho)^{1/2}$ exists, half the remaining frequencies will be lower and the other half higher. Hence $\omega = (m/\ell)(3E/\rho)^{1/2}$ is the middle one. Finally, the following immediate results can also be stated: 1) the upper bound on the highest frequency is 2 $(m/\ell)(3E/\rho)^{1/2}$; and 2) the lower half range of frequencies if upper-bounded by $(m/\ell)(3E/\rho)^{1/2}$.

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Effects of Density and Velocity Ratio on Discrete Hole Film Cooling

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Nomenclature

D = injection hole diameter
I = momentum flux ratio ($= \rho_j U_j^2 / \rho_\infty U_\infty^2$)
M = mass flux ratio ($= \rho_j U_j / \rho_\infty U_\infty$)
U = velocity
X = distance downstream from row of holes
ρ = density
η = effectiveness (= local surface mass fraction of foreign gas/mass fraction of foreign gas in injectant)

Subscripts

j = denotes the jet
∞ = denotes the mainstream

Introduction

CURRENT practice in the film cooling of gas turbine blades involves the use of rows of discrete holes for coolant injection. Much of the previous research on

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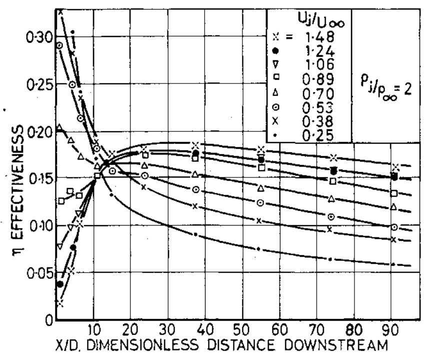

Fig. 1 Variation of centerline effectiveness with velocity ratio.

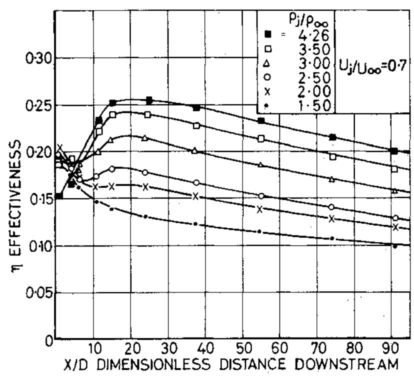

Fig. 2 Variation of centerline effectiveness with density ratio.

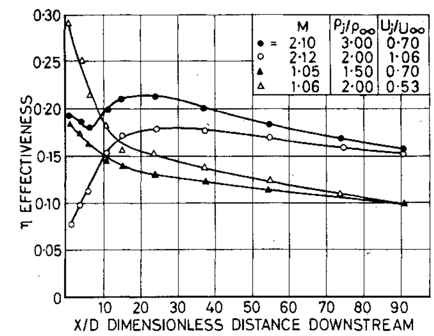

Fig. 3 Comparison of Figs. 1 and 2.

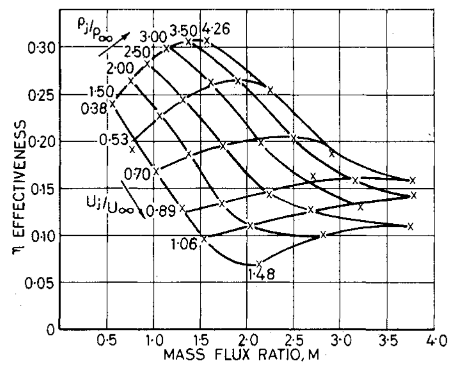

Fig. 4 Variation of effectiveness with mass flux ratio at 6.5 diam downstream.

discrete hole film cooling effectiveness has used a heated "coolant,"[1,2] giving injectant to mainstream density ratios of less than unity, whereas in practice this parameter will assume a value around 2.

Some work has been done on the effects of density ratio, notably by Goldstein et al.[3] and by Launder et al.[4]; however, the number of density ratios used has been small, so that no comprehensive information regarding the variation of effectiveness with density ratio could be obtained. In the present work, mixtures of air and Freon 12, a refrigerant gas, were injected into an airstream and the density ratio was varied continuously over the range 1.5-4.26, simply by changing the proportion of Freon 12 in the injectant. In addition, the ratio of injectant velocity to mainstream velocity was varied, so that the relative importance of these parameters could also be gaged.

For low-speed constant property flow, effectiveness is usually defined as the ratio of the difference between adiabatic wall and mainstream temperatures, to the difference between injectant and mainstream temperatures. In the present work use is made of the mass transfer analogy, in replacing the temperature differences by mass fractions. The use of the analogy is discussed by Goldstein, whose review paper[5] includes references to the work of a number of other investigators. For the analogy to hold, the primary requirement is that the turbulent Lewis number should be unity. The recent work of Nicolas and LeMeur[6] has shown this to be true for discrete hole injection.

Apparatus

A flat plate containing a row of holes was mounted in a traverse to form part of one wall of a rectangular duct measuring 381 mm × 133 mm. Air was supplied to the duct by a variable speed, tangential fan via a diffuser, gauze screens, and a contraction. The tunnel wall boundary layer was bled off upstream of the injection section, and a boundary-layer trip wire was incorporated to ensure a two-dimensional, turbulent boundary layer at the injection point. Downstream of the injection point, a row of sampling holes was sited on the tunnel centerline. The effectiveness was deduced from the mass fraction of Freon 12 in gas drawn slowly through these holes into a katharometer. The refrigerant gas and air passed through separate metering orifices before being thoroughly mixed and fed to the injection holes.

Test Conditions

In the tests reported here, the injection was normal to the mainstream through holes of 2.27 mm in diam and at 3 diam spacing. The mass flux ratio M was varied by varying the ratio of injectant to mainstream velocity from 0.38-1.48 over the range of density ratio of 1.5-4.26. In all cases the mainstream velocity was maintained at 30.5 m/sec and the ratio of the boundary-layer displacement thickness to the injection hole diameter was 0.62 at the injection point.

Results and Discussion

Data for both streamwise and spanwise variations of effectiveness were obtained, but only the streamwise data for the hole centerline are presented here. In Fig. 1 the variation of effectiveness is presented for varying velocity ratio at a constant density ratio of 2. Increasing the velocity ratio results in increased jet penetration and a corresponding reduction in effectiveness in the region of the injection holes. The higher injectant mass flux, produced by increasing the injectant velocity, produces a rise in effectiveness at distances in excess of 20 diam.

The results of Fig. 2 are for the same range of mass flux ratio, but with the injection velocity held constant and the density ratio varying from 1.5-4.26. Close to the hole, there is no strong dependence upon density ratio; while at distances in excess of 10 diam, the effectiveness again increases with injectant mass flux.

The curves in Figs. 1 and 2, although covering the same range of mass flux ratio, are strikingly different close to the injection holes. This is illustrated in Fig. 3, in which data for mass flux ratios of about 1 and 2 are replotted. Reducing the velocity ratio, while increasing the density ratio to keep the injectant mass flux constant, gives rise to greatly increased ef-

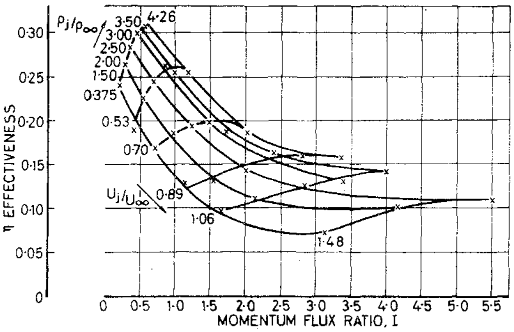

Fig. 5 Variation of effectiveness with momentum flux ratio at 6.5 diam downstream.

fectiveness close to the injection holes. This effect can also be seen in Fig. 4 where, at a fixed distance of 6.5 diam from the injection holes, the effectiveness is plotted for the full range of velocity and density ratios. Figure 5 shows that plotting the effectiveness against the momentum flux ratio I, as suggested by Goldstein,[3] does not collapse data at differing density ratios on those obtained at differing velocity ratios.

Conclusions

The results presented demonstrate that film cooling effectiveness is dependent upon the ratio of the density of the injectant to the mainstream, in addition to the velocity ratio. Further, it is apparent that account cannot be taken of the density ratio effect by working in terms of such simple parameters as mass flux ratio M or momentum flux parameter I. Until such time as a satisfactory correlating parameter has been found, it is recommended that testing of film cooling configurations be undertaken at density ratios as close as possible to those existing in practice.

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Nonexistence of Stationary Vortices Behind a Two-Dimensional Normal Plate

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THE problem is of the two-dimensional, irrotational flow of an inviscid incompressible fluid past an obstacle, with a pair of counter-rotating line-vortices lying symmetrically in the downstream flow, at rest relative to the body. Our interest in it arises from calculations we have made[1,2] of the three-dimensional flow over slender wings with leading-edge separation. If a wing has constant local span over a considerable part of its length, that is, it has a long parallel-sided region, we should expect the cross-flow in this region to be related to the two-dimensional flow of the title problem. We find that the leading-edge vortices continue to grow in strength and move upward away from the wing without apparent limit.

For the flow past a circular cylinder, Foppl showed that there is a locus of possible vortex positions, along which the circulation of the vortices varies.[3] This is what would be expected, since the vanishing of the two velocity components at the vortex positions imposes two conditions on the three unknowns, the two coordinates, and the circulation. If the obstacle is a flat plate, it is natural to impose a Kutta condition of finite velocity at the edge of the plate, and this condition enters in our calculations for slender wings. We should then expect that no more than a finite number of possible vortex positions would exist. It is surprising therefore, that both Riabouchinski[4] and Coe,[5] who introduce the Kutta condition, should again find loci for the vortex position. Our conclusion is quite different: we find no stationary vortex position behind a flat plate with the Kutta condition imposed.

It appears from a paper by Roy[6] that Villat[7] had come to the same conclusion in 1930. Since his work is not widely available and the algebra involved is not particularly lengthy, it seems worthwhile setting out the steps in full, in the hope of clarifying the situation.

We use Coe's notation in which the plate has width $2a$ and lies normal to a stream of speed U directed along the real axis of the Z-plane. Vortices of strength $\mp K$ (circulation $\mp 2\pi K$) which lie at $Z=Z_1$, and the conjugate point $\bar{Z}_1$, and the transformation $\zeta^2 = Z^2 + a^2$ is introduced. The vortex strength is positive for anticlockwise rotation. The Kutta condition is expressed by Coe's Eq. (4), which we write as

$$U = 2K\tau / (\sigma^2 + \tau^2) \tag{1}$$

by introducing σ and τ for the real and imaginary parts of $\zeta_1 = (Z_1{}^2 + a^2)^{1/2}$. The complex equation expressing the vanishing of the velocity at the position of the upper vortex is Coe's Eq. (6), which we write as

$$UZ_1 - KZ_1/2\tau + iKa^2/2Z_1\zeta_1 = 0 \tag{2}$$

When we divide this equation by Z_1 and take the imaginary part, we find

$$\Re\{Z_1{}^2\,\zeta_1\} = 0$$

Also

$$Z_1{}^2\zeta_1 = (\sigma^2 - \tau^2 - a^2 + 2i\sigma\tau)(\sigma + i\tau) \tag{3}$$

and so this condition gives

$$\sigma^2 - 3\tau^2 = a^2 \tag{4}$$

This is the same as Coe's Eq. (7) which he describes as the locus of the vortex. His Eq. (8) does not appear to follow from his Eq. (6). We find, on dividing Eq. (2) by Z_1 and now taking the real part,

$$U - K/2\tau = -Ka^2/\Im\{2Z_1{}^2\zeta_1\}$$

$$= -Ka^2/4\tau(\sigma^2+\tau^2), \text{ by (3) and (4)} \tag{5}$$

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